Solutions

3.2: Using Matrices to Solve Systems of Equations

Definition 1. A linear equation in n variables $x_1, x_2, \ldots, x_n$ is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, \ldots, a_n, b$ are constants. Again, the numbers $a_1, \ldots, a_n$ are the coefficients for the variables $x_1, \ldots, x_n$, respectively.

Remark 1: When the number of variables is small (≤ 3), we usually let $x_1 = x$, $x_2 = y$, and $x_3 = z$.

Definition 2. A matrix is a rectangular array of numbers. The augmented matrix of a system of linear equations is the matrix whose rows are the coefficient rows of the equations.

Example 1. Consider the system of equations

$$x + y = 3, \quad x - y = 1.$$

The augmented matrix of the system is given by

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \end{bmatrix}.$$

Elementary Row Operations. When dealing with an augmented matrix, the rows become representative of the equations themselves. So multiplying, adding or switching the orders of rows is the same as multiplying, adding or switching the orders of the equations they represent. In this way, we can use the augmented matrix to solve a system of equations.

Consider Example 1. Equations / Rows

E_1: $x+y=3$. R_1: $[1 \ 1 \ 3]$

E_2: $x-y=1$ R_2: $[1 \ -1 \ 1]$

Operations: (1) Multiplication. $aR_1: [a \ a \ 3a] \to aE_1: ax + ay = 3a$

(2) Addition/Subtraction. $R_1 + R_2: [2 \ 0 \ 4] \to E_1 + E_2: 2x = 4$

(3) Switch order. $R_1 \Leftrightarrow R_2 \to E_1 \Leftrightarrow E_2$.

Gauss-Jordan Decomposition:

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \end{pmatrix} \xrightarrow[R_2 - R_1]{R_2} \begin{pmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow[R_1 + \frac{1}{2}R_2]{R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -2 & -2 \end{pmatrix}.$$

Therefore $x = 2$ and $-2y = -2 \Rightarrow y = 1$.

Example 2. Use the Gauss-Jordan Reduction of the augmented matrix to solve the system

$$\begin{aligned} x - y + 5z &= -6 \\ 3x + 3y - z &= 10 \\ x + 3y + 2z &= 5. \end{aligned}$$

Augmented Matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right)$$

$R_2 \downarrow R_2 - 3R_1$

$R_3 \downarrow R_3 - R_1$

$$\left(\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right)$$

$R_2 \downarrow \frac{1}{2} R_2$

$$\left(\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 3 & -8 & 14 \\ 0 & 4 & -3 & 11 \end{array}\right)$$

$R_1 \downarrow 3R_1 + R_2$

$R_3 \downarrow 3R_3 - 4R_2$

$$\left(\begin{array}{ccc|c} 3 & 0 & 7 & -4 \\ 0 & 3 & -8 & 14 \\ 0 & 0 & 23 & -23 \end{array}\right)$$

$R_3 \downarrow \frac{1}{23} R_3$

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$$\left(\begin{array}{ccc|c} 3 & 0 & 7 & -4 \\ 0 & 3 & -8 & 14 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$R_1 \downarrow R_1 - 7R_3$

$R_2 \downarrow R_2 + 8R_3$

$$\left(\begin{array}{ccc|c} 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$R_1 \downarrow \frac{1}{3} R_1$

$R_2 \downarrow \frac{1}{3} R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Thus $x=1$, $y=2$, and $z=-1$.

The solution is

$$(x,y,z) = (1,2,-1).$$